BOUNDARIES ON SPACETIMES: AN OUTLINE

Steven G. Harris

Abstract. The causal boundary construction of Geroch, Kronheimer, and Penrose has some universal properties of importance for general studies of spacetimes, particularly when equipped with a topology derived from the causal structure. Properties of the causal boundary are detailed for spacetimes with spacelike boundaries, for multi-warped spacetimes, for static spacetimes, and for spacetimes with group actions.

0. Introduction

There is a deep history in mathematics of placing boundaries, sometimes thought of as ideal points, on mathematical objects which may not appear to come naturally equipped with boundaries. Perhaps the most famous example is the one-point compactification of the complex plane—i.e., the addition of a single "point at infinity"—resulting in the Riemann sphere. Often, there is more than one reasonable way to construct a boundary for a given object, depending on the intent; for instance, the plane—thought of as the real plane—is sometimes equipped, not with a single point at infinity, but with a circle at infinity, resulting in a space homeomorphic to a closed disk. Both these boundaries on the plane have useful, but different, things to tell us about the nature of the plane; the common feature is that, by bringing the infinite reach of the plane within the confines of a more finite object, we are better able to grasp the behavior of the original object.

The usefulness of the construction of boundaries for an object, in order to help realize behavior in the original object, has not been overlooked for spacetimes. The most common method of constructing a suitably illuminating boundary for a spacetime has been to embed it conformally in a larger spacetime (often termed an unphysical spacetime, in contrast with the original one, presumed to have more physical meaning), and then to use the boundary of the embedded image, with topology and causal properties induced from the ambient (unphysical) spacetime, as the boundary of the physical spacetime. This is the origin, for instance, of the usual picture of the boundary of Minkowski space—with future-timelike and past-timelike infinity i^+ and i^- (each a single point), future-null and past-null infinity $\mathcal{I}^+$ and $\mathcal{I}^-$ (each a null cone), and spacelike infinity i^0 (a single point)—derived from its standard embedding into the Einstein static spacetime (see, e.g., [HE]).

In 1972, Geroch, Penrose, and Kronheimer in [GKP] introduced a boundary-construction method for any strongly causal spacetime, a method which was conformally invariant, hence, a function only of the causal structure of the spacetime, insensitive to nuances of curvature and metric save in the grossest sense. They called the resulting boundary the causal boundary. The importance of the causal

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boundary seemed perhaps more to lie in the very general nature of its construction and its apparent naturality, rather than in any particular insights derived from the application of this method to specific spacetimes. This is because the construction was rather involved with a topology that could be quite opaque in even simple cases. (None the less, the causal boundary was used for good effect in understanding the nature of the two-dimensional trousers spacetime in [HD].)

A series of papers in recent years ([Uni], [Top], [Stat]; [Grp] in progress) has attempted to demonstrate both the specific utility of the causal boundary in a categorical sense of universality and methods of explicating the causal boundary for wide classes of spacetimes. This note will summarize the progress made to date and explore some possibilities still being investigated.

Section 1 explores one of the motivations for believing that the nature of the boundary of a spacetime is of importance in understanding the global structure of a spacetime: the possibility that invariance of spatial topology may be related to the causal nature of the causal boundary.

Section 2 outlines the construction of the causal boundary and of a topology different from that considered by Geroch, Kronheimer, and Penrose: what might be called the future chronological topology. The universality of the future causal boundary construction, in terms of category theory, is detailed.

Section 3 details the universality of the future chronological topology for the future causal boundary, within the limited category of spacetimes with spacelike future boundaries. A simple class of examples is that of multi-warped spacetimes with spacelike boundaries.

Section 4 shows how to construct the causal boundary for any standard static spacetime. These are spacetimes with, essentially, product geometries, $\mathbb{L}^1 \times M$ for M Riemannian. Such spacetimes have null boundaries.

Section 5 looks at the effect on the causal boundary of forming a quotient of a spacetime by a group action. This includes more general static spacetimes.

Section 6 briefly looks at some other recent work on spacetime boundaries and mentions questions for future investigation.

1. Topology Change and Boundaries

One of the persistent questions in cosmology is whether the spatial topology of the universe is constant in time. Just what is meant by "spatial topology of the universe" is not entirely clear, absent some very specific structure being assumed *a priori* for the spacetime. For general purposes, one can allow any suitable spacelike hypersurface in a spacetime to be an exemplum of spatial topology; the question then becomes, do any two such have the same topology?

To be more specific: "Suitable" here probably should mean a hypersurface which is embedded, achronal, and edgeless in some sense. Achronal means not just spacelike, but that, further, no two points be timelike-related. "Edgeless" can have a number of reasonable meanings; perhaps the simplest is that the embedding be a proper map (a sequence of points in the domain is convergent if and only if the image sequence is convergent in the ambient space). Then the question of invariance of spatial topology can be formulated thus:

If a spacetime V contains two spacelike hypersurfaces M_1 and M_2, both edgeless and achronally embedded, is it necessarily true that M_1 and M_2 are homeomorphic?

It is not hard to show that this is true for V being Minkowski space, $\mathbb{L}^n$, and

that, in fact, any such hypersurface must be diffeomorphic to $\mathbb{R}^{n-1}$; a proof appears in [Min]. This is also true in the more general case of a standard static spacetime $V = \mathbb{L}^1 \times N$ for N any Riemannian manifold; any achronal, edgelessly embedded spacelike hypersurface must be diffeomorphic to N. The proof of Theorem 3 in [GH] suffices for this. (That is a far more general theorem containing, in addition, the assumption that V be timelike or null geodesically complete, which amounts in this case to assuming that N be complete; but the proof works without that assumption for the simplified setting of a standard static spacetime.)

The usual boundary for Minkowski space $\mathbb{L}^n = \mathbb{L}^1 \times \mathbb{R}^{n-1}$—for instance, the boundary of its image under the standard conformal embedding into the Einstein static spacetime, $\mathbb{L}^1 \times \mathbb{S}^{n-1}$ (see [HE])—consists of a null cone in the future and a null cone in the past (cones on the boundary sphere $\mathbb{S}^{n-2}$ of $\mathbb{R}^{n-1}$). For the standard static spacetime $V = \mathbb{L}^1 \times N$, in case N is complete, the causal boundary of V is much like a null cone, a cone on a kind of boundary at infinity on N, though the exact nature of the topology is a subtle issue. In case N is not complete, the causal boundary of V is still cone-like, but some of the cone elements are timelike.

Theorem 3 in [GH] actually establishes invariance of spatial topology in the much broader context of any stationary spacetime (i.e., possessing a timelike Killing field) which is timelike or null geodesically complete and obeys the chronology condition (no closed timelike curves): Any achronal, edgelessly embedded spacelike hypersurface must be diffeomorphic to the space of Killing orbits. This is generalized in [HL] (Theorem 4.3, supplemented by Theorem 2 of [Mth]) to any strongly causal spacetime V possessing a foliation $\mathcal{F}$ by timelike curves such that every 2-sheet $S \subset V$ ruled by $\mathcal{F}$ has the property that, thought of as a spacetime in its own right, every ruling γ in S enters the past and the future of every point of S: Any achronal, edgelessly embedded spacetime in such a V must be diffeomorphic to the leafspace of $\mathcal{F}$.

The causal boundary for such general spacetimes as in the paragraph above is far from clear. But for a static spacetime (having a timelike Killing field which is hypersurface-orthogonal) which is chronological and geodesically complete, the causal boundary is again somewhat akin to a null cone over an appropriate object.

But it is very easy to come up with simple spacetimes which do not exhibit invariance of spatial topology. A well-known example is de Sitter space, $\mathbb{D}^n = \{p \in \mathbb{L}^{n+1} \,|\, p \text{ is unit-spacelike}\}$; $\mathbb{D}^n$ has Cauchy surfaces which are $\mathbb{S}^{n-1}$ but also edgeless, achronal, spacelike hypersurfaces which are $\mathbb{R}^{n-1}$. This is actually an example (up to conformal factor) of the more general setting of a product static spacetime $V = I \times N$ with I an interval of $\mathbb{L}^1$ which is finite at one or both ends. For instance, let $I = (-\infty, 0)$. Then V clearly has edgeless, achronal, spacelike hypersurfaces diffeomorphic to N, such as $\{t\} \times N$ for any $t < 0$. But consider any map $f : N \to \mathbb{R}$ which obeys $|\mathrm{grad}(f)| < 1$, and let $N^- = \{x \in N \,|\, f(x) < 0\}$. Then the map $\phi : N^- \to V$ defined by $\phi : x \mapsto (f(x), x)$ is an edgeless, achronal, spacelike embedding; with dimension of N at least two, we can always choose f so that N^- has a different topology from that of N.

The boundary for de Sitter space $\mathbb{D}^n$, in its conformal mapping into the Einstein static spacetime, is a spacelike $\mathbb{S}^{n-1}$ for the future and the same again for the past. The causal boundary for $(-\infty, 0) \times N$, if N is complete, is a spacelike N for the future (with something like a null cone for the past).

These examples motivate the following notion:

Vague Conjecture. *If a spacetime has a causal boundary which has a substantial spacelike nature, then it does not exhibit invariance of spatial topology. If it has a causal boundary which is much like a null cone, both in the future and in the past, then it does have invariance of spatial topology.*

This is the sort of idea that suggests there is probably much merit in learning the structure of the causal boundary of as many spacetimes as possible.

2. Constructions

a) Basics and Causal Structure.

The central idea of the causal boundary of Geroch, Kronheimer, and Penrose is to construct an endpoint for every endless timelike curve, in such a way that the future endpoint of a curve γ depends only on the curve's past $I^-[\gamma]$, and its past endpoint depends only on its future $I^+[\gamma]$; two future-endless timelike curves should share the same constructed future endpoint if and only if they have the same past, and similarly for past-endless. (Square brackets are employed here for a set-function defined on points to denote its extension to sets; thus, $I^-[A]$ means $\bigcup\{I^-(x) \,|\, x \in A\}$, where $I^-(x)$ is the usual past of a point, $I^-(x) = \{y \,|\, y \ll x\}$. The $\ll$ relation is the usual chronology relation in a time-oriented spacetime, $y \ll x$ meaning that there is a timelike curve from y to x, future-directed in that order.)

In the sequel, all constructions will be assumed also to be defined in the time-dual manner, *mutatis mutandis*.

The means of construction for the future causal boundary—the future endpoints of future-endless timelike curves—is to work with indecomposable past sets, known as IPs. A set P is a past set if $I^-[P] = P$; it is an indecomposable past set if it cannot be expressed as the union of two proper past subsets. It turns out (see, for instance, [HE]) that there are exactly two kinds of IPs in a spacetime: the past of any point $I^-(x)$ and the past of any future-endless timelike curve $I^-[\gamma]$; the former are sometimes called PIPs, the latter TIPs (for point-like and terminal IPs). We can then define the future causal boundary $\hat{\partial}(V)$ of a spacetime V to be, quite simply, the TIPs of V: $\hat{\partial}(V) = \{P \,|\, P \text{ is an IP and for all } x \in V, P \neq I^-(x)\}$.

It is crucial that we get not just a set for the future causal boundary but that there be an extension of the chronology relation from V to $\widehat{V} = V \cup \hat{\partial}(V)$ (which may be called the future completion of V). The GKP construction does this in a unified manner for the entire future completion at once; but in many spacetimes this introduces some new chronology relations between the points of V. This expanded notion of chronology within V itself is of importance, but it is possible to extend $\ll$ in V to $\widehat{V}$ without introducing any expansion within the spacetime proper, and only later to make the expanded definition; and that is the procedure followed here.

So this is the (simple) extension of $\ll$ from V to $\widehat{V}$:

(1) For $x \in V$ and $P \in \hat{\partial}(V)$, $x \ll P$ iff $x \in V$.
(2) For $x \in V$ and $P \in \hat{\partial}(V)$, $P \ll x$ iff $P \subset I^-(w)$ for some $w \ll x$ ($w \in V$).
(3) For $P, Q \in \hat{\partial}(V)$, $P \ll Q$ iff $P \subset I^-(w)$ for some $w \ll x$ ($w \in V$).

The expanded notion of $\ll$, called here the past-determined chronology and denoted by $\ll^p$, is defined by including all the relations above, all pairs $x \ll y$ in V, and also $x \ll^p y$ if $I^-(x) \subset I^-(w)$ for some $w \ll y$ for $x, y, w \in V$. Call a spacetime past-determined if $\ll^p = \ll$; this includes globally hyperbolic spacetimes and warped products of Riemannian manifolds with past-determined spacetimes.

The causality relation on V—$x \prec y$ means there is a causal curve from x to y, future-directed in that order—also extends to $\widehat{V}$, via $x \prec V$ iff $I^-(x) \subset P$, $P \prec x$ iff $P \subset I^-(x)$, and $P \prec Q$ iff $P \subset Q$. This is not used in any of the categorical notions.

The GKP construction continues by defining the dual notion of the past causal boundary $\check{\partial}(V)$ and then detailing a very elaborate scheme for making identifications among elements of the past and future causal boundaries, resulting in a topology for the combination of V with the equivalence classes of boundary points; details can be seen in [HE]. There have been some modifications suggested for the identification scheme, such as in [BS] and, notably, in [S].

But there are inherent problems in this combined approach, of treating future and past causal boundary elements together. Very troubling, for instance, is the fact that the topology generated for the boundary of Minkowski space is not that of its conformal embedding into the Einstein static spacetime: In the GKP topology, each cone-element (a null line) is an open set in the boundary. Further, there are spacetimes for which the combined future-and-past causal boundary is neither future- nor past-complete, in an appropriate sense. Hence, the approach I have followed is to deal solely with the future causal boundary, as $\widehat{V}$ is future-complete in a strong sense, and the future-completion construction has important universal properties.

To see the universality, we must greatly extend the applicability of the future completion process (details in [Uni]). It turns out that one doesn't need much structure to apply the completion process: only a set X with a relation $\ll$ (called the chronology relation) such that $\ll$ is transitive and non-reflexive ($x \not\ll x$), there are no isolates (everything is related chronologically to something), and X has a countable set S which is dense: if $x \ll y$ then for some $s \in S$, $x \ll s \ll y$. Call this a chronological set. The role of timelike curves in a chronological set is played by future chains: sequences $\{x_n\}$ obeying $x_1 \ll \cdots \ll x_n \ll x_{n+1} \ll \cdots$. The role of a future endpoint to a timelike curve is played by the notion of future limit to a future chain $c = \{x_n\}$; this is a point x such that $I^-(x) = I^-[c]$. This will be unique if X is past-distinguishing (i.e., $I^-(x) = I^-(y)$ implies $x = y$), such as in a strongly causal spacetime or its future completion.

An indecomposable past set in a chronological set X can be defined exactly as in a spacetime, and $P \subset X$ is an IP if and only if $P = I^-[c]$ for some future chain c. Then the future causal boundary of X is defined as before, $\hat{\partial}(X) = \{P \subset X \mid P \text{ is an IP and for all } x \in X, P \neq I^-(x)\}$. The $\ll$ relation extends to $\widehat{X} = X \cup \hat{\partial}(X)$ exactly as in a spacetime, and also the past-determined expansion, $\ll^p$.

In this broader picture, $\widehat{X}$ is the same sort of creature as X: a chronological set. Furthermore, the identification of $\widehat{X}$ as "the future completion" of X can be made explicit in this sense: Call a chronological set future-complete if every future chain has a future limit (not necessarily unique); then $\widehat{X}$ is always future-complete. Why it should be called *the* future-completion of X (actually, it is $\widehat{X}$ with the past-determined expansion that deserves this title) lies in the categorical nature of the enterprise.

The basic category of interest is that of **ChronologicalSet**: The objects are chronological sets and the morphisms are set-functions which preserve the chronology relation and future limits: $x \ll y$ implies $f(x) \ll f(y)$, and x is a future limit

of the future chain c implies $f(x)$ is a future limit of the future chain $f[c]$; call such a function future-continuous. **ChronologicalSet** has various full subcategories obtained by restricting to future-complete, past-distinguishing, or past-determined chronological sets.

Within the subcategory of past-determined and past-distinguishing objects, the process of future-completion is a functor into the future-complete subcategory; that is to say, for any future-continuous $f : X \to Y$ between past-determined, past-distinguishing chronological sets, there is a future-continuous map $\hat{f} : \widehat{X} \to \widehat{Y}$, and this construction preserves composition of functions ($\widehat{X}$ and $\widehat{Y}$ are both past-determined and past-distinguishing, as well as future-complete). Indeed, $\hat{f}$ is just the "natural" extension of f: The injections $\hat{\imath}_X : X \to \widehat{X}$ form a natural transformation from the identity functor to the future-completion functor, and $\hat{f} \circ \hat{\imath}_X = \hat{\imath}_Y \circ f$. Finally, if Y is already future-complete, than $\hat{f} : \widehat{X} \to Y$ is the unique future-continuous map obeying $\hat{f} \circ \hat{\imath}_X = f$.

This means precisely that the future-completion functor and the $\hat{\imath}$ natural transformation form a left adjoint to the forgetful functor from future-complete, past-determined, past-distinguishing chronological sets to past-determined, past-distinguishing ones. Then by category theory (see, for instance, [M]) we know that this functor and natural transformation are unique, up to natural equivalence, in providing a functorial way of future-completing the category of past-determined, past-distinguishing chronological sets. In effect, $\widehat{X}$ is the unique minimal future-completing object for X within the category of past-determined, past-distinguishing chronological sets.

The restriction to past-determined objects is annoying, as one wishes to consider spacetimes which are not past-determined; if there are "holes" in a spacetime, one still expects to be able to apply a causal boundary construction, hoping to fill in those holes. This is where the expansion to $\ll^p$ comes in. This, too, can be expressed in a functorial, natural, and universal manner to an appropriate subcategory of chronological sets; we just need to restrict attention to chronological sets which are "past-regular": $I^-(x)$ is an IP for every point x. As every spacetime is past-regular, this is a reasonable restriction.

Performing past-determination and following with future-completion thus yields a functorial, natural, and universal construction for forming a future-complete (and past-determined) chronological set from a past-regular, past-distinguishing one (and preserving the latter qualities). But this is not quite the GKP construction: That requires first doing future-completion and then performing the past-determination expansion. But it turns out that the two ways of ordering are naturally equivalent. Thus, the GKP future causal boundary construction provides the minimal—categorically unique—way of future-completing a past-regular, past-distinguishing chronological set, retaining those qualities.

So what about the full GKP causal boundary, compounded out of the past and future boundaries? For any chronological set X, one can extend the chronology relation not just to $\widehat{X} = X \cup \hat{\partial}(X)$, but also to $\bar{X} = X \cup \hat{\partial}(X) \cup \check{\partial}(X)$ (that last being the past causal boundary). An appropriate equivalence relation $\cong$ on $\hat{\partial}(X) \cup \check{\partial}(X)$ may allow for a chronology relation on the quotient $X^* = \bar{X}/\cong$. This is often not past-regular, as some of the elements of $\hat{\partial}(X)$ may have become identified.

Failure of past-regularity is not necessarily a bad thing; when this occurs for a

spacetime V (example: remove a finite timelike segment from $\mathbb{L}^2$), it is likely enough that V^* is the desired completion object, and not $\widehat{V}$. It may not be future-complete, but it is likely to have a sort of generalized notion of future completeness; this generalized future completeness is intended to take note of the situation of a future chain having a future limit in $\widehat{V}$, but that future limit being identified with other boundary elements in V^*. If it does have this generalized future completeness (not all spacetimes do; [Uni] contains an example which does not), then upon deleting the elements of V^* that have no past, the remainder is a quotient of $\widehat{V}$. This has a generalization in categorical terms: The uniqueness of the extension to $\widehat{X}$ of a future-continuous map from X into a future-complete Y is replicable for the appropriate generalized notions derived from identifications.

b) Topology.

As shown in [Top], the topology imputed to the causal boundary of Minkowski space in [GKP] does not at all match the common expectation of a cone, such as one obtains by taking the inherited topology from conformally embedding Minkowski space in the Einstein static spacetime. Modifications of the GKP process, such as in [BS] and [S], have no different effect in such a simple spacetime.

The process presented here, summarizing the results of [Top], is applicable for any past-regular, past-distinguishing chronological set. This topology seems to do the right thing in a number of contexts. It has especially good qualities (such as universality) in the category of chronological sets with spacelike boundaries, as described in section 3. This topology can be generalized to deal with non-past-regular chronological sets, also, but that will not be detailed here (it appears in [Top]).

The heart of the topology to be defined on a chronological set X is not any notion of open set, but of what may be called a limit-operator on sequences in X: If $\sigma = \{x_n\}$ is a sequence of points in X, then $L(\sigma)$ is the set of "first-order" limits of σ. Under normal circumstances, one expects $L(\sigma)$ to be either empty or have exactly one element; but the future chronological topology on X might be non-Hausdorff (and this is even physically reasonable, if $X = \widehat{V}$ for a spacetime V in which some elements of $\hat{\partial}(V)$ ought perhaps to be identified), in which case some sequences will have more than one first-order limit. In any case, we get a topology from L so long as it does the right thing on subsequences: For every subsequence $\tau \subset \sigma$, we must have $L(\tau) \supset L(\sigma)$. Then the L-topology is defined by declaring that a set A is closed if and only if for every sequence σ in A, $L(\sigma) \subset A$.

(The reason for calling the elements of $L(\sigma)$ first-order limits is that $L(\sigma)$ might be infinite, and for some sequence $\tau \subset L(\sigma)$, there may be elements of $L(\tau)$—second-order limits of σ—not appearing in $L(\sigma)$. For chronological sets, it takes a highly unusual one for second-order limits to exist; in any case, this has no effect on the development of the ideas here.)

For the future chronological topology on a past-regular chronological set, the limit-operator $\hat{L}$ is defined thus: Let $\sigma = \{x_n\}$; then $x \in \hat{L}(\sigma)$ if and only if

(1) for all $y \ll x$, for sufficiently large n, $y \ll x_n$, and
(2) for any IP $P \supset I^-(x)$, if $P \neq I^-(x)$, then for some $y \in P$, for sufficiently large n, $y \not\ll x_n$.

This can be formulated in terms of set-limits: For any sequence of sets $\{A_n\}$, let

$$\mathrm{LI}(\{A_n\}) = \lim_{n\to\infty}\inf(\{A_n\}) = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k$$
$$\mathrm{LS}(\{A_n\}) = \lim_{n\to\infty}\sup(\{A_n\}) = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k$$

i.e., $x \in \mathrm{LI}(\{A_n\})$ if and only if $x \in A_n$ for n sufficiently large, and $x \in \mathrm{LS}(\{A_n\})$ if and only if $x \in A_n$ for infinitely many n; clearly, $\mathrm{LI}(\{A_n\}) \subset \mathrm{LS}(\{A_n\})$. If the A_n are all past sets in a chronological set, then $\mathrm{LI}(\{A_n\})$ and $\mathrm{LS}(\{A_n\})$ are past sets, also. Then for a sequence $\sigma = \{x_n\}$ in a past-regular chronological set X, $x \in \hat{L}(\sigma)$ if and only if

(1) $I^-(x) \subset \mathrm{LI}(\{I^-(x_n)\})$ and
(2) $I^-(x)$ is a maximal IP within $\mathrm{LS}(\{I^-(x_n)\}$.

In particular, if $\mathrm{LI}(\{I^-(x_n)\}) = \mathrm{LS}(\{I^-(x_n)\})$, then in $\widehat{X}$, $\hat{L}(\sigma)$ is non-empty, as every past set contains at least one maximal IP.[1]

There are a number of important points to note about this construction:

(1) The $\hat{L}$-topology—the future chronological topology—is not just a topology for the future causal boundary added to a spacetime. Rather, it is a topology that can be defined in any chronological set (even past-regularity can be dispensed with); thus, a spacetime supplemented with any sort of boundary at all, compounded with an extension of the chronology relation to the boundary, may be fitted with the future chronological topology.

(2) The future chronological topology fits well with the notion of a future limit of a future chain: For any future chain c, $\hat{L}(c)$ is precisely the set of future limits of c. It follows that in a past-distinguishing chronological set, a future chain has at most one topological limit in this topology.

(3) The future chronological topology on a strongly spacetime V is precisely the same as the manifold topology on V.

(4) This topology respects (future) boundary constructions generally: Suppose X is a past-regular chronological set and X is a subset of $\bar{X}$, with the chronology relation on X extending to $\bar{X}$, making $\bar{X}$ also a past-regular chronological set; further, suppose that X is chronologically dense in $\bar{X}$, i.e., for all $a, b \in \bar{X}$ with $a \ll b$, there is some $x \in X$ with $a \ll x \ll b$. Then the topology induced on X as a subspace of $\bar{X}$ (with its $\hat{L}_{\bar{X}}$-topology) is the same as the $\hat{L}_X$-topology on X, treated as a chronological set in its own right; and X is topologically dense in $\bar{X}$.

Combining points (2), (3), and (4), we see that for V a strongly causal spacetime, using the future chronological topology for $\widehat{V}$ yields the elements of $\hat{\partial}(V)$ as topological endpoints for future-endless timelike curves.

(5) If V is a strongly causal spacetime, then in the $\hat{L}$-topology on $\widehat{V}$, $\hat{\partial}(V)$ is a closed subset.

(6) If $V = \mathbb{L}^n$, then the $\hat{L}$-topology on $\widehat{V}$ is precisely the same as that given by the conformal embedding into the Einstein static spacetime, $\mathbb{L}^1 \times \mathbb{S}^{n-1}$. In other words: If $\phi : V = \mathbb{L}^n \to E = \mathbb{L}^1 \times \mathbb{S}^{n-1}$ is the conformal embedding, then

[1] This formulation of $\hat{L}$ is due to J. L. Flores.

$\hat{\phi} : \widehat{V} \to \widehat{E}$ is a homeomorphism onto its image. In particular, the $\hat{L}$-topology of $\hat{\partial}(\mathbb{L}^n)$ is that of a cone on $\mathbb{S}^{n-2}$.

For a past-regular, past-distinguishing chronological set X, there is not much choice when looking for a future-complete, past-distinguishing boundary. Although there can be some play with respect to past-determination, that has no effect on the future chronological topology. Any past-distinguishing future completion of X must be homeomorphic to $\hat{X}$ in the $\hat{L}$-topology. (But non-past-distinguishing future completions can be more desirable.)

In the generalization of future chronological topology to non-past-regular chronological sets, points (1), (2), and (4) above remain true. But the strongest topological results require spacelike boundaries.

3. Spacelike Boundaries

The categorical and universality results for **ChronologicalSet** mentioned in section 2.a are notably absent in section 2.b. There are, indeed, universal results in a topological category; but we must restrict ourselves to chronological sets with spacelike boundaries.

The problem is that if $f : X \to Y$ is a future-continuous map between past-determined chronological sets, which is continuous in the respective $\hat{L}$-topologies, on X and Y, then even though $\hat{f} : \widehat{X} \to \widehat{Y}$ is future-continuous, it is not necessarily continuous in the $\hat{L}$-topologies on $\widehat{X}$ and $\widehat{Y}$. In other words: Even with f in the topological category, $\hat{f}$ may not be, thus destroying the functoriality of future completion. Examples of this can be found for such simple spacetimes as $Y = \mathbb{L}^2$ and $X = \{(x,t) \in \mathbb{L}^2 \,|\, x > 0\}$; for some continuous functions $f : X \to Y$, the extension of f to the boundary on X is necessarily discontinuous, and this can be done for f preserving $\ll$. But this cannot happen when the boundary is spacelike.

In a past-regular chronological set X, call a point x inobservable if $I^-(x)$ is a maximal IP: no IP properly contains it. Then we will say that X has only spacelike boundaries (more properly: only spacelike future boundaries) if (a) all elements of $\hat{\partial}(X)$ are inobservable (in $\widehat{X}$) and (b) $\hat{\partial}(X)$ is closed in $\widehat{X}$ or the set of inobservables of $\widehat{X}$ form a closed subset in $\widehat{X}$. (The reason for (b) is purely technical; it is satisfied in all reasonable instances, such as $\widehat{V}$ for V a strongly causal spacetime.) A future-continuous map $f : X \to Y$ between past-regular, past-distinguishing chronological sets is said to preserve spacelike boundaries if $\hat{f}$ preserves inobservables. Then the category of interest is **FutureTopology SpacelikeBoundaries PastRegular PastDistinguishing ChronologicalSet**: objects are past-regular, past-distinguishing chronological sets with only spacelike boundaries, and morphisms are future-continuous maps that are continuous in the respective future chronological topologies and that preserve spacelike boundaries.

The important result is that discontinuity can arise only on timelike and null boundaries: If f is a morphism in the future topological category, then so is $\hat{f}$. As all the injections $\hat{\imath}_X$ are in this category when X is, it follows that the categorical results from section 2.a apply also to the topological category above: Future completion is a functor into the future-complete subcategory, and future completion together with the $\hat{\imath}$ injections form a left adjoint to the forgetful functor.

Perhaps the most interesting results lie in the category of the generalizations for non-past-regular chronological spaces. All the categorical and universality results

apply, and there is also a form of topological semi-rigidity:

Semi-rigidity of future completion. *A generalized future completion of X consists of a map $i : X \to Y$ with i an isomorphism of $\ll$ onto its image $Y_0 = i[X]$, Y satisfying the generalized notion of future-complete, and every point of $\partial(Y) = Y - Y_0$ being a generalized future limit of a future chain in Y_0.*

Suppose Y is a generalized future completion of X, where X and Y have only spacelike boundaries, obey the generalization of past-distinguishing, and have no points with empty pasts. Then in the $\hat{L}$-topologies, Y is a topological quotient of $\widehat{X}$. Furthermore, if X is past-regular, then $\partial(Y)$ is a topological quotient of $\hat{\partial}(X)$ (more generally, $\partial(Y)$ is a topological quotient of a closely related structure).

In particular: If V is a strongly causal spacetime with only spacelike boundaries, then any generalized future-completing boundary for V (in other words, anything reasonably called a sort of future completion) is a topological quotient of $\hat{\partial}(V)$, in the $\hat{L}$-topologies.

A common way of providing a boundary for a spacetime is to embed it into another spacetime and consider the boundary of the image. Actually, this can be done with a topological embedding into any manifold. If V is our spacetime and $\phi : V \to N$ is a map into a manifold N such that ϕ is a homeomorphism onto its image, then for any $p \in N$, define $I^-_V(p)$ to be those points $x \in V$ such that there is a future-directed timelike curve from x to p (i.e., its ϕ-image approaches p). Then we can consider the ϕ-future-boundary of V, $\partial^+_\phi(V)$, to consist of those points $p \in N$ with $I^-_V(p) \neq \emptyset$. The ϕ-future-completion of V, V^+_ϕ, consists of $V \cup \partial^+_\phi(V)$, topologized by identifying V with its image $\phi[V]$ in N. We obtain a causal structure on V^+_ϕ in a manner similar to that for $\widehat{V}$: For $x \in V$ and $p, q \in \partial^+_\phi(V)$,

(1) $x \ll p$ if and only if $x \in I^-_V(p)$
(2) $p \ll x$ if and only if for some $y \ll x$ ($y \in V$), $I^-_V(p) \subset I^-(y)$
(3) $p \ll q$ if and only if for some $y \in I^-_V(q)$, $I^-_V(p) \subset I^-(y)$

This will always yield the structure of a chronological set for V^+_ϕ, which therefore has a future chronological topology, as well as the topology induced by ϕ. How do these two topologies compare? That depends on whether V has only spacelike boundaries and on how ϕ extends to $\widehat{V}$.

Embedding topology and future chronological topology. *Let V be a strongly causal spacetime with only spacelike boundaries. Suppose $\phi : V \to N$ is a topological embedding of V into a manifold N and that ϕ extends continuously to a map $\bar{\phi} : \widehat{V} \to N$.*

(1) *If $\bar{\phi}$ is a proper map onto its image, then the future chronological topology on V^+_ϕ is the same as the ϕ-induced topology.*
(2) *If the restriction of $\bar{\phi}$ to $\partial^+_\phi(V)$ is a proper map onto its image, then the future chronological topology on $\partial^+_\phi(V)$ (from V^+_ϕ) is the same as the ϕ-induced topology.*

There is a class of spacetimes that lend themselves to calculation of the causal boundary, so that we can make a judgement as to whether the future chronological topology yields something reasonable. This is the class of multi-warped spacetimes

(or anything conformal to such): As a manifold, V is the topological product $(a,b)\times K_1 \times \cdots \times K_m$ for manifolds $K_1,\ldots,K_m$ (a and b can be finite or infinite). For each i, K_i has a Riemannian metric h_i and there is a function $f_i : (a,b) \to \mathbb{R}^+$. The metric on V is $-(dt)^2 + f_1(t)h_1 + \cdots f_m(t)h_m$. Examples (up to conformal factor) include

(1) interior Schwarzschild: $V = (0,2m)\times\mathbb{R}^1\times\mathbb{S}^2$ with

$$(\frac{2m}{r}-1)(ds)^2 = -(dr)^2 + (\frac{2m}{r}-1)^2(dt)^2 + r^2(\frac{2m}{r}-1)h_{\mathbb{S}^2}$$

where $h_{\mathbb{S}^2}$ is the usual metric on the 2-sphere

(2) Robertson-Walker spacetimes: $V = (a,b)\times K$ with

$$(ds)^2 = -(dt)^2 + r(t)^2 h$$

where (K,h) is a quotient of $\mathbb{S}^3$, $\mathbb{R}^3$, or $\mathbb{H}^3$, and $r(t)$ is the characteristic length for the universe at time t

(3) generalized Kasner spacetimes: $V = (0,\infty)\times\mathbb{R}^1\times\mathbb{R}^1\times\mathbb{R}^1$ with

$$(ds)^2 = -(dt)^2 + t^{2a}(dx)^2 + t^{2b}(dy)^2 + t^{2c}(dz)^2$$

where a,b,c are constants

A multi-warped spacetime has only spacelike future boundaries if and only if for each i, the Riemannian metric h_i is complete and the warping function grows sufficiently large at the future end of the interval: Assuming that b is the future end, the required condition is that $\int_{b^-}^{b} f_i^{-1/2} < \infty$ for any finite $b^- < b$.

Thus, interior Schwarzschild has only spacelike future boundaries (0 is the future end of the interval). A Robertson-Walker space will have only spacelike future boundaries if and only if $\int_{b_-}^{b} 1/r(t) < \infty$, which is precisely the condition that the spacetime be conformal to a finite-in-the-future portion of the standard static spacetime $\mathbb{L}^1\times K$. A generalized Kasner spacetime has only spacelike future boundaries if and only if the Kasner exponents a,b,c are > 1.[2]

Multi-warped spacetimes. *Let $V = (a,b)\times K_1\times\cdots\times K_m$ be a multi-warped spacetime, with b the future end of the interval; let $K = K_1\times\cdots\times K_m$.*

If V has only spacelike future boundaries, then in the future chronological topology, $\widehat{V} \cong (a,b]\times K$ with $\hat{\partial}(V)\cong K$ (appearing as $\{b\}\times K$).

Combining these three points: Any "reasonable" future-completing boundary for interior Schwarzschild is (in the future chronological topology) a topological quotient of $\mathbb{R}^1\times\mathbb{S}^2$. This includes anything derived from a topological embedding which extends to a continuous and proper map on the future causal boundary, using the topology induced by the embedding.

[2] This was stated incorrectly in [Uni], but correctly given in [GS].

4. Standard Static Spacetimes

We will consider a construction that yields the future causal boundary for any standard static spacetime—or anything conformal to such. The details appear in [Stat].

A static spacetime is one with a timelike Killing field U such that $U^\perp$ is integrable. A standard static spacetime is one of the form $\mathbb{R}^1 \times M$ with metric $g = -\Omega(dt)^2 + h$, for some positive function $\Omega : M \to \mathbb{R}^+$ and some Riemannian metric h on M. Because the causal boundary depends only on the conformal class of the spacetime (being defined purely in terms of the causal structure and topology), we might as well confine ourselves to standard static spacetimes with $\Omega \equiv 1$ (as g above is conformal to $-(dt)^2 + (1/\Omega)h$). Thus, we will look at a spacetime which is a product, $V = \mathbb{L}^1 \times M$ for M any Riemannian manifold.

Past sets are easy to characterize in V. For any function $f : M \to \mathbb{R}$, define $\mathrm{P}(f) = \{(t,x) \,|\, t < f(x)\}$. Then $\mathrm{P}(f)$ is a past set if and only if f is a Lipschitz-1 function (i.e., $|f(x) - f(y)| < d(x,y)$, where d is the distance function on M); and every past set arises in this manner, save for the past set which is V itself—which can be represented as $\mathrm{P}(\infty)$.

The IPs of V are sets of the form $\mathrm{P}(f)$ for particularly special Lipschitz-1 functions: what might be called the Busemann functions on M. These come about as follows:

Every IP P is generated as the past of a timelike curve σ in V, which may be parametrized as $\sigma(t) = (t, c(t))$ for some curve c in M satisfying $|\dot c(t)| < 1$ ($|v|$ denotes the length of a vector v in M using the Riemannian metric there). Let $[\alpha, \omega)$ be the domain of σ in this parametrization (with the possibility that $\omega = \infty$, but with α finite), hence, the domain of c. Then a calculation yields that $P = I^-[\sigma] = \mathrm{P}(b_c)$, where $b_c : M \to \mathbb{R}$ is given by $b_c(x) = \lim_{t\to\omega}(t - d(c(t), x))$. Actually, b_c may be infinite-valued; but either $b_c(x) = \infty$ for all $x \in M$, or b_c is finite-valued on all of M; and in the latter case, it is Lipschitz-1. If $P = I^-((s,x))$, then $\omega = s$, $x = \lim_{t\to s} c(t)$, and $b_c = d^s_x : y \mapsto s - d(x,y)$. The future causal boundary of V consists of all the other IPs: $\mathrm{P}(\infty)$ (which we can call i^+ in imitation of $\mathbb{L}^n$) and $\mathrm{P}(b_c)$ for finite b_c which is not any d^s_x, i.e., such that $c(t)$ has no limit as $t \to \omega$.

If c is a minimizing unit-speed geodesic, then the function b_c is precisely the Busemann function for c, such as is used in the construction of the boundary sphere for a Hadamard manifold: simply connected, non-positive-curvature, complete Riemannian manifold (see [BGS], for example). Accordingly, we may call b_c the Busemann function for c, even for a non-geodesic curve c. (We could just as easily restrict our attention to unit-speed c here, instead of less-than-unit-speed, as the only difference is looking at null curves instead of timelike curves to generate IPs, and either is satisfactory for these spacetimes. But we cannot ignore non-geodesic curves, as there are curves c such that $b_c \neq b_\gamma$ for any geodesic γ, and $\mathrm{P}(b_c)$ is part of $\hat\partial(V)$.)

Let $\mathcal{L}_1(M)$ denote the Lipschitz-1 functions on M, and let $\mathcal{B}(M)$ be the finite Busemann functions which are not any d^s_x, i.e., functions of the form b_c for a curve $c : [\alpha, \omega) \to M$ with no limit-point at ω. Within $\mathcal{B}(M)$, b_c is bounded if and only if c has finite length, which is equivalent to $\omega < \infty$ (since $|\dot c| \leq 1$); let $\mathcal{B}_{\mathrm{finite}}(M)$ be these Busemann functions, $\mathcal{B}_\infty(M)$ the rest. If M is complete, then $\mathcal{B}_{\mathrm{finite}}(M)$ is empty.

So we can identify $\hat\partial(V)$ with $\mathcal{B}(M) \cup \{i+\}$, with a splitting of $\mathcal{B}(M)$ into "finite"

and "infinite" parts. But what about topology?

We can map V into $\mathcal{L}_1(M)$ by sending (s,x) to d_x^s and look for a boundary of V inside $\mathcal{L}_1(M)$. The natural topology on $\mathcal{L}_1(M)$ is as a function space, using the compact-open topology; the function-space topology is quite simple when restricted to $\mathcal{L}_1(M)$ as convergence in that topology is the same as point-wise convergence of the functions. This mapping is a topological embedding with the function-space topology on $\mathcal{L}_1(M)$, and it is tempting to use this topology for $V \cup \mathcal{B}(M)$ inside $\mathcal{L}_1(M)$ (identifying V with its image).

There is a real action on $\mathcal{L}_1(M)$, with $a \cdot f$ (for $a \in \mathbb{R}$ and $f \in \mathcal{L}_1(M)$) defined by $a \cdot f : x \mapsto f(x) + a$. This action preserves $\mathcal{B}(M)$ ($a \cdot b_c = b_{c^a}$ for $c^a(t) = c(t-a)$) and is reflected in V by $a \cdot (s,x) = (s+a,x)$. As $V/\mathbb{R} = M$, one can look to $\mathcal{L}_1(M)/\mathbb{R}$ for various boundaries on M. For instance, since $\mathcal{L}_1(M)/\mathbb{R}$ is compact (using the function-space topology on $\mathcal{L}_1(M)$), one can achieve a compactification of M by looking for its closure in $\mathcal{L}_1(M)/\mathbb{R}$; one might call the boundary of M obtained thereby the Lipschitz boundary of M, $\partial_{\mathrm{Lip}}(M)$. Let us call $\mathcal{B}(M)/\mathbb{R}$ the Busemann boundary of M, $\partial_{\mathrm{Bus}}(M)$ (whether using the function-space topology from $\mathcal{L}_1(M)$ or some other topology). While $\partial_{\mathrm{Lip}}(M)$ has some claim on us as a natural boundary on M, it is $\partial_{\mathrm{Bus}}(M)$ that is central to the causal boundary on $\mathbb{L}^1 \times M$.

Let $\partial_{\mathrm{Bus}}^{\mathrm{finite}}(M) = \mathcal{B}_{\mathrm{finite}}(M)/\mathbb{R}$ and $\partial_{\mathrm{Bus}}^{\infty}(M) = \mathcal{B}_{\infty}(M)/\mathbb{R}$, as those subsets of $\mathcal{B}(M)$ are also preserved by the $\mathbb{R}$-action. Then $\partial_{\mathrm{Bus}}^{\mathrm{finite}}(M)$ represents the Cauchy completion of M, while the elements of $\partial_{\mathrm{Bus}}^{\infty}(M)$ can be said to represent the points at "geometric infinity" for M, as they derive from curves which either are rays (half-infinite geodesics minimizing along the entire length) or behave asymptotically like rays. If M is a Hadamard manifold, then $\partial_{\mathrm{Bus}}(M)$ (using the function-space topology) is precisely the boundary sphere for M; but in the general setting, even for complete M, $\partial_{\mathrm{Bus}}(M)$ need not be compact or anything like a manifold.

Let $\pi : \mathcal{L}_1(M) \to \mathcal{L}_1(M)/\mathbb{R}$ denote the projection to the quotient. For each point $\beta \in \partial_{\mathrm{Bus}}(M)$, $\pi^{-1}(\beta)$ is, of course, a line. If $\beta \in \partial_{\mathrm{Bus}}^{\mathrm{finite}}(M)$, then it is a line of timelike-related elements in $\hat{\partial}(V)$, while if $\beta \in \partial_{\mathrm{Bus}}^{\infty}(M)$, then it is a line of null-related elements; that is to say, for $a > 0$, $\mathrm{P}(b_c) \subset \mathrm{P}(a \cdot b_c)$, and if $b_c \in \mathcal{B}_{\mathrm{finite}}(M)$, then $\mathrm{P}(b_c) \ll \mathrm{P}(a \cdot b_c)$. There are no timelike relations among the "infinite" elements, though there may be other null relations. In particular, for M complete, there are only null relations within the future causal boundary.

In the function-space topology, any choice of $x_0 \in M$ yields an evaluation map $e : \mathcal{L}_1(M) \to \mathbb{R}$, $e : f \mapsto f(x_0)$, which is continuous. This yields a continuous cross-section $z : \mathcal{L}_1(M)/\mathbb{R} \to \mathcal{L}_1(M)$ given by $z : [f] \mapsto f - e(f)$. The same cross-section works for $\pi : \widehat{V} - \{i^+\} \to M \cup \partial_{\mathrm{Bus}}(M)$ and $\pi : \hat{\partial}(V) - \{i^+\} \to \partial_{\mathrm{Bus}}(M)$. Since a fibre-bundle with a cross-section is a trivial bundle (i.e., a product), this means that $\widehat{V}$ and $\hat{\partial}(V)$, apart from i^+, are products. Adding in i^+, we obtain that $\hat{\partial}(V)$ is a cone on $\partial_{\mathrm{Bus}}(M)$—a null cone, if M is complete (though there may be some other null relations than those obtaining along each cone element).

But that is with the function-space topology imputed to $\hat{\partial}(V)$; and that topology may not be the future chronological topology. Functions which converge in the function-space topology (point-wise convergence) always converge in the future chronological topology, but there may also be convergence in the $\hat{L}$-topology which is not point-wise—and although $\hat{\partial}(V) - \{i^+\} \to \partial_{\mathrm{Bus}}(M)$ and $\widehat{V} - \{i^+\} \to M \cup \partial_{\mathrm{Bus}}(M)$ are still fibre bundles, the $\hat{L}$-convergence can destroy the continuity of

the evaluation map e, allowing the fibre bundles to be non-trivial.[3] This is likely to happen when M has significant amounts of positive curvature going out to infinity, so that there are geodesics that are not minimizing.

An example of such M is the plane with the region between $y = R$ and $y = -R$ ($R > 1$) being changed to have uniform positive curvature $1/R^2$ (think of the universal cover of a grapefruit impaled by a stick).[4] In the flat plane, all half-lines generate elements of $\mathcal{B}(M)$, with parallel half-lines generating the same elements of $\partial_{\text{Bus}}(M)$; thus, $\partial_{\text{Bus}}(M)$ is a circle at infinity. But with the roundness inserted, the geodesics $c(t) = (t, a)$ (or $(-t, a)$) have $b_c = \infty$ for $|a| < R$; thus, $\partial_{\text{Bus}}(M)$ consists of two arcs. Let b^+ be the Busemann function for $c^+(t) = (t, 2R)$ and b^- be that for $c^-(t) = (t, -2R)$. Then b^+ and b^- are intertwined in an interesting manner, and $\pi(b^+)$ and $\pi(b^-)$ are not Hausdorff-separated in $M \cup \partial_{\text{Bus}}(M)$ if we use the future chronological topology. Let σ be the sequence in $V = \mathbb{L}^1 \times M$ consisting of $\{(n, Rn, 0)\}$. Then σ has no limit in $\widehat{V}$ using the function-space topology from $\mathcal{L}_1(M)$: σ converges to an element of $\mathcal{L}_1(M)$, but it is not in $\mathcal{B}(M)$ (this is an example of $\partial_{\text{Bus}}(M)$ being smaller than $\partial_{\text{Lip}}(M)$). But in the future chronological topology, σ converges to both b^+ and b^-, and $\widehat{V} - \{i^+\}$ is not a product over $M \cup \partial_{\text{Bus}}(M)$.

The Busemann boundary can be complicated to work out in detail, but its overall features are often fairly clear. For instance, $\partial_{\text{Bus}}(\mathbb{R}^n) = \partial_{\text{Bus}}(\mathbb{H}^n) = \mathbb{S}^{n-1}$. If K is compact, then $\partial_{\text{Bus}}(K) = \emptyset$ and $\partial_{\text{Bus}}(N \times K) = \partial_{\text{Bus}}(N)$; and more generally, $\partial_{\text{Bus}}(N_1 \times N_2)$ can be expressed as a sort of product involving $\partial_{\text{Bus}}(N_1)$ and $\partial_{\text{Bus}}(N_2)$. The Busemann boundary works well with connected sum: $\partial_{\text{Bus}}(N_1 \# N_2) = \partial_{\text{Bus}}(N_1) \dot\cup \partial_{\text{Bus}}(N_2)$ (disjoint union).

In summary:

Structure of the future causal boundary for a standard static spacetime. *For $V = \mathbb{L}^1 \times M$, $\hat\partial(V)$ consists of i^+ and a set of other elements that are organized as null lines and (if M is not complete) timelike lines, all joined to i^+. Aside from i^+, $\widehat{V}$ has a free real action extending the obvious one on V, yielding a line bundle of $\hat\partial(V) - \{i^+\}$ over $\partial_{\text{Bus}}(M)$ and of $\widehat{V} - \{i^+\}$ over $M \cup \partial_{\text{Bus}}(M)$. In the function-space topology, these bundles are trivial, yielding product structures for $\widehat{V}$ and $\hat\partial(V)$ aside from $\{i^+\}$; hence, $\hat\partial(V)$ is a cone on $\partial_{\text{Bus}}(M)$. In the future chronological topology, that product structure may not obtain, though one might still consider $\hat\partial(V)$ to be cone-like.*

5. Group Actions

Oftentimes, a spacetime V of complicated topology is easier to analyze by looking at its universal cover, $\widetilde{V}$, which has the advantage of being simply connected. For instance, if V is a static-complete spacetime (i.e., possesses a complete, hypersurface-orthogonal, timelike Killing field), then $\widetilde{V}$ is a standard static spacetime, conformal to a product $\mathbb{L}^1 \times M$ (see, for instance, theorem 4 in [GH]). Since we already know a lot about how to find the causal boundary of standard static spacetimes, we may hope to use that to get information on the boundary of the original spacetime.

[3] This corrects a misstatement in [Stat].

[4] Some of the analysis of this space is due to J. L. Flores.

The relation between $\widetilde{V}$ and V is that there is a group G which acts on $\widetilde{V}$, and $V = \widetilde{V}/G$; of course, $G = \pi_1(V)$, the fundamental group of V. So we are led to the question of how to derive information on the boundary of a quotient of a spacetime by a group action. More generally, we can inquire into the boundary of the quotient of a chronological set X by a group G which acts on X, whose action preserves the chronology relation, and which yields a chronological set for the quotient X/G: How is $\hat{\partial}(X/G)$ related to structures in X and $\widehat{X}$? This is explored in [Grp].

First, what sort of group action on a chronological set X yields a chronological set for the quotient X/G? This is very simple: We plainly want each group element $g \in G$ to induce a chronological isomorphism on X (i.e., $x \ll y$ implies $g \cdot x \ll g \cdot y$). Then X/G will be a chronological set under the relation $[x] \ll [y]$ whenever $x \ll g \cdot y$ for some $g \in G$ (with $[x]$ denoting the equivalence class of x), if and only if for all $g \in G$ and $x \in X$, $x \not\ll g \cdot x$; if this holds, call it a chronological group action. When the group action is chronological, the projection $\pi : X \to X/G$ ($\pi : x \mapsto [x]$) is future-continuous.

Note that there are two possible topologies to consider for X/G: the quotient topology (using the future chronological topology on X) and the future chronological topology, considering X/G as a chronological set in its own right. Naturally, π is continuous with the quotient topology on X/G; but it may well not be with the future chronological topology on X/G. If X and X/G are strongly causal spacetimes, then we know that the two topologies on X/G are the same, because both are the manifold topology; but the interesting question is with X being the future completion of a spacetime.

As might be expected, the G action on X extends to $\widehat{X}$, so we may consider $\widehat{X}/G$ and $\hat{\partial}(X)/G$. One might hope that there is some simple relation between these objects and $\widehat{X/G}$ and $\hat{\partial}(X/G)$, but this is generally not the case, even for very simple examples. As an instructive example, consider $X = \mathbb{L}^2$ and $G = \mathbb{Z}$, the integers, with the action $m \cdot (t, x) = (t, x + m)$; then $X/G = (\mathbb{L}^1 \times \mathbb{R}^1)/\mathbb{Z} = \mathbb{L}^1 \times (\mathbb{R}^1/\mathbb{Z}) = \mathbb{L}^1 \times \mathbb{S}^1$, the Minkowski cylinder.

We have $\hat{\partial}(\mathbb{L}^2) = \{i^+\} \cup \{P^a_L \,|\, a \in \mathbb{R}\} \cup \{P^a_R \,|\, a \in \mathbb{R}\}$, where $i^+ = \mathbb{L}^2$, $P^a_L = \{(t, x) \,|\, t < -x + a\}$, and $P^a_R = \{(t, x) \,|\, t < x + a\}$. The $\mathbb{Z}$ action extends to the boundary by $m \cdot i^+ = i^+$, $m \cdot P^a_L = P^{a+m}_L$, and $m \cdot P^a_R = P^{a-m}_R$. The topology of $\hat{\partial}(\mathbb{L}^2)$ is that of a cone on $\mathbb{S}^0$ (the 0-sphere, two points), i.e., a line. The $\mathbb{Z}$-action is not free, and the quotient topology on $\hat{\partial}(\mathbb{L}^2)/\mathbb{Z}$ is quite nasty: Each of the two null portions of the boundary is rolled up into a circle, while the image of i^+ in that quotient is a viciously non-Hausdorff point, whose only neighborhood is all of those two circles. If we look at $\widehat{\mathbb{L}^2}/\mathbb{Z}$ as a chronological set and take its future chronological topology, the imposed topology on $\hat{\partial}(\mathbb{L}^2)/\mathbb{Z}$ isn't any better: It's the wholly indiscrete topology.

The boundary on $\mathbb{L}^2/\mathbb{Z}$ is easy to work out from the material in section 4: $\hat{\partial}(\mathbb{L}^1 \times \mathbb{S}^1)$ is a null cone on $\partial_{\mathrm{Bus}}(\mathbb{S}^1)$; since $\mathbb{S}^1$ is compact, its Busemann boundary is empty. Therefore, $\hat{\partial}(\mathbb{L}^2/\mathbb{Z})$ is just the single point $\{i^+\}$ and not $\hat{\partial}(\mathbb{L}^2)/\mathbb{Z}$ in either topology.

On the other hand, consider the same $\mathbb{Z}$-action on lower Minkowski space, ${\mathbb{L}^2}_- = \{(t, x) \,|\, t < 0\}$. We have $\hat{\partial}({\mathbb{L}^2}_-) = \{P^a \,|\, a \in \mathbb{R}\}$, where $P^a = \{(t, x) \,|\, t < |x - a|\}$, and the $\mathbb{Z}$-action extends to the boundary by $m \cdot P^a = P^{a+m}$. Thus, the quotient is a circle, $\hat{\partial}({\mathbb{L}^2}_-)/\mathbb{Z} = \mathbb{S}^1$, and there is only one topology: The quotient topology

on $\widehat{\mathbb{L}^2{}_-/\mathbb{Z}}$ is the same as its topology as a chronological set in its own right. And the other way around is the same thing: $\mathbb{L}^2{}_-/\mathbb{Z}$ is the lower half of the Minkowski cylinder, $(-\infty, 0) \times \mathbb{S}^1$, and the material from section 3 shows its future boundary to be $\mathbb{S}^1$. In this instance, $\hat{\partial}(X/G) = \hat{\partial}(X)/G$ and $\widehat{X/G} = \widehat{X}/G$, and the two topologies, quotient and future chronological, coincide on $\widehat{X}/G$.

So if $\hat{\partial}(X/G)$ is not generally $\hat{\partial}(X)/G$, what is it, and how do we find it? The answer lies in considering the invariant sets in X that project to IPs and boundary elements in X/G. Define a group-indecomposable past set (or GIP) in X to be a G-invariant past set which is not the union of two proper subsets which are G-invariant past sets. These are precisely the sets of the form $\bigcup(G \cdot P)$ where P is an IP (the notation here is $G \cdot x = \{g \cdot x \,|\, g \in G\}$ and $G \cdot A = \{G \cdot a \,|\, a \in A\}$); π maps every GIP onto an IP, and the inverse image of every IP is a GIP. Define the G-future causal boundary of X to be $\hat{\partial}_G(X) = \{A \subset X \,|\, A \text{ is a GIP and for all } x \in X, A \neq \bigcup(G \cdot I^-(x))\}$. Then for any GIP A in X, $A \in \hat{\partial}_G(X)$ if and only if $\pi[A] \in \hat{\partial}(X/G)$; similarly, for an IP P in X/G, $P \in \hat{\partial}(X/G)$ if and only if $\pi^{-1}[P] \in \hat{\partial}_G(X)$. We can define a bijection $\hat{\pi}^\partial : \hat{\partial}_G(X) \to \hat{\partial}(X/G)$ via $\hat{\pi}^\partial : A \mapsto \pi[A]$, and this can be a very handy way of determining the elements of $\hat{\partial}(X/G)$.

But we wish to understand the topology of $\hat{\partial}(X/G)$ (in terms of structures in X), and not just what its elements are. One way to do this is to realize the GIPs of X as IPs of another chronological set. This can be done by defining a new relation on X: set $x \ll_G y$ if and only if $x \ll g \cdot y$ for some $g \in G$. Then $X_G = (X, \ll_G)$ is a chronological set, called the G-expansion of X. It is a very strange chronological set, being massively non-past-distinguishing; but it has the nice property that the IPs of X_G are precisely the GIPs of X, and $\hat{\partial}(X_G) = \hat{\partial}_G(X)$. Furthermore, X_G captures all the right topological information (despite the fact that X_G itself is massively non-Hausdorff), in the following sense: Let $\pi_G : X_G \to X/G$ be the same as π on the set-level; this is continuous in the respective $\hat{L}$-topologies, if X is past-regular (which implies that X/G is, also). Then π_G extends to $\widehat{\pi_G} : \widehat{X_G} \to \widehat{X/G}$, also continuous. This map takes the boundary to the boundary, so we can consider the restriction $\widehat{\pi_G}^\partial : \hat{\partial}(X_G) \to \hat{\partial}(X/G)$; on the set-level, this is the same as the map $\hat{\pi}^\partial : \hat{\partial}_G(X) \to \hat{\partial}(X/G)$. The pay-off is this:

Topology of the future causal boundary of X/G**.** *If X is a past-regular chronological set with a chronological action from a set G, then $\hat{\partial}(X/G)$ can be identified with $\hat{\partial}_G(X)$ or $\hat{\partial}(X_G)$; in the respective future chronological topologies on X_G and X/G, the map $\widehat{\pi_G}^\partial : \hat{\partial}(X_G) \to \hat{\partial}(X/G)$ is a homeomorphism.*

Furthermore, the attachment of $\hat{\partial}(X/G)$ to X/G is exactly reflected in the attachment of $\hat{\partial}_G(X)$ to X—interpreted as $\hat{\partial}(X_G)$ attaching to X_G—via the map $\widehat{\pi_G} : \widehat{X_G} \to \widehat{X/G}$, in that a sequence σ in $\widehat{X_G}$ converges to an element $A \in \hat{\partial}(X_G)$ if and only if $\widehat{\pi_G}[\sigma]$ converges to $\widehat{\pi_G}(A)$ in X/G.

Here is a typical application:

Let V be a chronological static-complete spacetime. In [GH] it is shown that the space M of Killing orbits is a manifold ($\Pi : V \to M$ is a line bundle), the universal cover $\widetilde{V}$ is conformal to the standard static spacetime $\mathbb{L}^1 \times \widetilde{M}$, where $\widetilde{M}$ is the space of Killing orbits in $\widetilde{V}$, and the universal covering map $\pi_V : \widetilde{V} \to V$ induces a map $\pi_M : \widetilde{M} \to M$ which is the universal covering map for M. Let $G = \pi_1(V)$, which

is also $\pi_1(M)$, so that π_V is the projection $\widetilde{V} \to \widetilde{V}/G$, and π_M is the projection $\widetilde{M} \to \widetilde{M}/G$. The G-action on $\widetilde{V}$ splits into the action of G on $\widetilde{M}$ and a linear action on $\mathbb{L}^1$, i.e., a group homomorphism $\mu : G \to \mathbb{R}$; in other words, the action of G on $\widetilde{V} = \mathbb{L}^1 \times \widetilde{M}$ is given by $g \cdot (t, p) = (t + \mu(g), g \cdot p)$. The homomorphism μ can be detected in V as follows: Pick any loop c in V representing the element $g \in G = \pi_1(V)$; then

$$\mu(g) = \int_c \alpha, \text{ where } \alpha = -\frac{\langle -, U\rangle}{|U|^2}$$

for U the Killing field and $\langle -, -\rangle$ the metric in V. In fact, the metric is $-|U|^2\alpha^2 + \Pi^* h$, where h is a Riemannian metric on M. We have the restriction (from the chronology condition on V) that for all $g \neq e$ (the identity element) and all $p \in \widetilde{M}$, $|\mu(g)| < d(p, g\cdot p)$, where d is the distance function from h. Another way to interpret μ is as an element of $H^1_{\mathrm{dR}}(V)$, the first de Rahm cohomology group for V; it is a fundamental algebraic invariant of the static spacetime.

From section 4, we already know how to find the boundary IPs in $\widetilde{V} = \mathbb{L}^1 \times \widetilde{M}$. To find the boundary IPs of V, we just need to discover the G-invariant items.

For a concrete example, suppose we know that the space of Killing orbits is a Möbius strip crossed with $\mathbb{R}^1$, i.e., $M = \mathbb{R}^3/\mathbb{Z}$ with the action $m \cdot (x, y, z) = (x + m, (-1)^m y, z)$. Then $\widetilde{V}$ is conformal to the product $\mathbb{L}^1 \times \mathbb{R}^3$, and the map $\pi_V : \widetilde{V} \to V$ is projection by the $\mathbb{Z}$-action $m\cdot(t, x, y, z) = (t+\mu m, x+m, (-1)^m y, z)$, where μ is some real number with $|\mu| < 1$. The elements of the future causal boundary of $\widetilde{V}$, apart from i^+, are all IPs of the form $P^a_{\mathbf{u}} = \{(t, p) \,|\, t < a + \langle p, \mathbf{u}\rangle\}$, where $\mathbf{u}$ is a unit vector. For $\mathbf{u} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$, we have $m \cdot P^a_{\mathbf{u}} = P^{a+(\mu-\alpha)m}_{\mathbf{u}'}$, where $\mathbf{u}' = \mathbf{u}$ if m is even, and $\mathbf{u}' = \bar{\mathbf{u}} = \alpha\mathbf{i} - \beta\mathbf{j} + \gamma\mathbf{k}$ if m is odd. In particular, we see that $\cup(\mathbb{Z} \cdot P^a_{\mathbf{u}}) = i^+$ unless $\alpha = \mu$. Thus, the boundary GIPs, apart from i^+, are $\{Q^a_{\beta,\gamma} \,|\, a \in \mathbb{R}, \beta^2 + \gamma^2 = 1 - \mu^2\}$, where $Q^a_{\beta,\gamma} = P^a_{\mathbf{u}} \cup P^a_{\bar{\mathbf{u}}}$ for $\mathbf{u} = \mu\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$.

In other words, $\hat{\partial}(V)$ is a null cone on $\mathbb{S}^1/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by reflection across the y-axis. Convergence to elements of this boundary is found by looking for convergence in the pre-images in the future completion of the $\mathbb{Z}$-expansion of $\mathbb{L}^1 \times \mathbb{R}^3$.

So when is it that $\hat{\partial}(X/G) = \hat{\partial}(X)/G$ or $\widehat{X/G} = \widehat{X}/G$? And when is it that the quotient and future chronological boundaries on $\widehat{X}/G$ are the same? The answer lies with spacelike boundaries:

Group actions with spacelike boundaries. *Suppose V is a spacetime with a group G acting chronologically, freely, and properly discontinuously such that*

(1) *V/G is strongly causal (which forces V to be also) and*
(2) *V and V/G both have only spacelike boundaries.*

Then the quotient and future chronological topologies are the same on $\widehat{V}/G$. If, in addition,

(3) *$\widehat{V}/G$ is past-distinguishing,*

then $\widehat{V/G}$ is homeomorphic to $\widehat{V}/G$, and $\hat{\partial}(V/G)$ is homeomorphic to $\hat{\partial}(V)/G$.

Condition (3) is unfortunate, in that it is fairly opaque. It amounts to saying that if the G-orbit of a boundary IP P covers another boundary IP Q, then Q must

be an element of the G-orbit of P; this is not something that is particularly easy to check. Nor is it at all clear that this needs to be stated as a hypothesis, as no examples have appeared to date that satisfy (1) and (2) but not (3); perhaps (3) can be derived from the others.

For multi-warped spacetimes, (3) comes for free (as the IPs are all very obvious):

Quotients of multi-warped spacetimes with spacelike boundaries. *Suppose V is a strongly causal spacetime with only spacelike boundaries which is covered, by means of a G-action, by a multi-warped spacetime W, also with only spacelike boundaries. Then $\widehat{V} \cong \widehat{W}/G$ (in either topology, as they are the same), and $\hat{\partial}(V) \cong \hat{\partial}(W)/G$.*

6. Other Work and Future Research

Other authors have done work in recent years in exploration of boundary concepts. One particularly intriguing idea has been developed recently by Marolf and Ross: a re-examination of the entire causal boundary, combining future and past boundaries in a new manner; this is detailed in [MR]. Although they express their idea for strongly causal spacetimes, much of it works just as well for chronological sets. It can be presented thus:

For X a past- and future-regular chronological set, define the Szabados relation (it first appeared in [S]) between the IPs and IFs of X as follows: For P an IP and F an IF, $P \backsim_{\mathrm{Sz}} F$ if and only if (1) $F \subset \bigcap\{I^+(x) \,|\, x \in P\}$ and F is a maximal IF in that intersection, and (2) $P \subset \bigcap\{I^-(x) \,|\, x \in F\}$ and P is a maximal IP in that intersection. For instance, for any $x \in X$, $I^-(x) \backsim_{\mathrm{Sz}} I^+(x)$ (and that is the only Szabados-related pair involving $I^-(x)$ or $I^+(x)$). Szabados used this relation, extended for transitivity, to define an equivalence relation on $\hat{\partial}(X) \cup \check{\partial}(X)$, as a modification of the GKP procedure. Marolf and Ross, however, have an entirely different use for this relation (without any extension). They define a boundary for X as

$$\begin{aligned}\bar{\partial}(X) =& \{(P, F) \,|\, P \in \hat{\partial}(X), F \in \check{\partial}(X), P \backsim_{\mathrm{Sz}} F\} \cup \\ & \{(P, \emptyset) \,|\, P \in \hat{\partial}(X) \text{ and for all } F \in \check{\partial}(X), P \not\backsim_{\mathrm{Sz}} F\} \cup \\ & \{(\emptyset, F) \,|\, F \in \check{\partial}(X) \text{ and for all } P \in \hat{\partial}(X), P \not\backsim_{\mathrm{Sz}} F\}.\end{aligned}$$

Then the MR completion of X is $\bar{X} = X \cup \bar{\partial}(X)$. Treating any point $x \in X$ as the pair $I^{\pm}(x) = (I^-(x), I^+(x))$, so that $\bar{X}$ can be looked at as a set of ordered pairs, a chronology relation on $\bar{X}$ is defined by $(P, F) \ll (P', F')$ if and only if $F \cap P' \neq \emptyset$. This makes $\bar{X}$ into a chronological set, and the embedding of X into $\bar{X}$ as $x \mapsto I^{\pm}(x)$ is a chronological isomorphism onto its image: No new chronological relations are introduced into X (as opposed to the GKP construction, which ends with a past- and future-determined chronological set).

A few points of note on the chronological issues: X is chronologically dense in $\bar{X}$, which is past- and future-regular. Unless X has only spacelike future boundaries, $\bar{X}$ is not necessarily past-distinguishing, but it is always past/future-distinguishing (a point is determined by its past and its future together). It is past- and future-complete, though there may be more than one future limit to a future chain or past limit to a past chain: For a future chain c, any $(I^-[c], F)$ with $I^-[c] \backsim_{\mathrm{Sz}} F$ (or $F = \emptyset$ if there is no such IF) is a future limit for c.

Marolf and Ross define a topology for $\bar{X}$, bearing some similarity in method to the future- and past-chronological topologies. For V a strongly causal spacetime, the MR topology on $\bar{V}$ induces the manifold topology on V, which is topologically dense in $\bar{V}$; $\bar{\partial}(V)$ is closed. A future limit for a timelike curve is a topological endpoint (though there may be more than one such). Not only is $\bar{V}$ not necessarily Hausdorff (which may be expected), but it might not even be T_1: points might not be closed. If V has only spacelike future boundaries then we can compare the future chronological topology on $\widehat{V}$ with the MR topology by mapping $\widehat{V}$ into $\bar{V}$ via $P \mapsto (P, \emptyset)$ for $P \in \hat{\partial}(V)$; if $\widehat{V}$ is Hausdorff—as seems to be generally likely—this is a topological embedding onto the image (so the same result for multi-warped spacetimes obtains as in section 3). For a standard static spacetime $V = \mathbb{L}^1 \times M$ with M complete, the same mapping works (as $P \backsim_{\mathrm{Sz}} F$ is impossible); the MR topology in this case is the same as the function-space topology on $\widehat{V}$. Thus, in particular, for $V = \mathbb{L}^n$ the MR construction gives the same topology on $\bar{V}$ as the embedding into the Einstein static spacetime.

An approach to boundaries with an eye towards classification of boundary points by singularity-type is the thrust of Scott and Szekeres in [SS]. This is an examination of boundaries formed by topological embeddings. Starting with a spacetime (or, indeed, any manifold) V, they define an envelopment of V to be a smooth topological embedding $\phi : V \to W$ into another manifold of the same dimension, with open image. For an envelopment ϕ, let $\partial_\phi(V)$ be the boundary of $\phi(V)$ in the target space. A boundary set for V is then any subset of such a boundary. The key notion in [SS] is that of an equivalence relation among boundary sets for V: If $B \subset \partial_\phi(V)$ and $B' \subset \partial_{\phi'}(V)$ are boundary sets for V, then $B \simeq B'$ means that for all sequences of points $\{p_n\}$ in V, $\{\phi(p_n)\}$ approaches a limit in B if and only if $\{\phi'(p_n)\}$ approaches a limit in B'. An abstract boundary point for V is then defined to be any equivalence class of boundary sets for V which contains a singleton set $\{p\}$ as one of the elements of the class. The abstract boundary for V, $\partial_{\mathrm{ab}}(V)$, is the set of all abstract boundary points for V. Thus, each element of $\partial_{\mathrm{ab}}(V)$ can be represented by a point in some $\partial_\phi(V)$, but there are always larger boundary sets, using different envelopments, which are equivalent in terms of approach by points in V.

The main thrust of Scott and Szekeres is a schemata for classification of abstract boundary points in terms of being regularizable, points at infinity, or singularities. Being a point at infinity has to do with being approachability by curves in V of some particular class $\mathcal{C}(V)$, such as geodesics or curves of bounded acceleration. It must be a class that can be divided into subclasses of finite and of infinite parameter-length, independent of allowable change in parametrization. Different choices for $\mathcal{C}(V)$ may yield different classifications. Those abstract boundary points which can be approached only by the curves in $\mathcal{C}(V)$ of infinite parameter-length are points at infinity; singularities are approachable by at least one curve of finite parameter-length.

García-Parrado and Senovilla have introduced a study of boundaries from envelopments (in the sense of [SS]) informed by another kind of equivalence relation, which they call isocausality; this is detailed in [GS]. They call two spacetimes V and V' isocausal, written $V \sim V'$, if there exist diffeomorphisms $\phi : V \to V'$ and

$\psi : V' \to V$, both of which preserve the chronology relation. Then a causal extension of a spacetime V is an envelopment $\Phi : V \to W$ such that $V \sim \Phi[V]$. They present a classification of points in $\partial_\Phi(V)$ as singularities and points at timelike or spacelike infinity. They present numerous detailed examples, including negative-mass Schwarzschild, Reissner-Nordström, generalized Kasner, and other Bianchi-I spacetimes.

Finally, some directions for future research:

Preliminary work with J. L. Flores suggests that it is possible to develop a time-symmetric chronological topology, utilizing both $\hat{L}$ and $\check{L}$ (i.e., the limit-operators for the future and past chronological topologies). How useful is this? How does it compare with the MR topology on the MR completion of a spacetime?

It is possible to develop the Scott-Szekeres approach in the direction of chronological sets by introducing a chronology relation on an appropriate subset of the abstract boundary, such as those abstract boundary points approachable by future-directed timelike curves in the spacetime V (i.e., extending the notion of $\partial^+_\phi(V)$ in section 3 from a single envelopment to abstract boundary points); call this the future abstract boundary, $\hat{\partial}_{\mathrm{ab}}(V)$. One can then consider the future chronological topology on $V \cup \hat{\partial}_{\mathrm{ab}}(V)$. As the abstract boundary is a very unwieldy set, this may make it possible to understand it a bit better, though it's unclear what questions it would answer.

For V a standard static spacetime $\mathbb{L}^1 \times M$, when is it that $\widehat{V}$ is a simple product (aside from i^+) over the Busemann completion of M? And when is the function-space topology the same as the future chronological topology for $\widehat{V}$? (These appear to be the same question.) Joint work with Flores is currently aimed at trying to characterize Riemannian manifolds M for which this holds.

If V is a chronological static-complete Riemannian manifold, then, as shown in section 5, for M the space of Killing orbits, $\widetilde{V} = \mathbb{L}^1 \times \widetilde{M}$, and there are a G-action on $\widetilde{M}$ (with $G = \pi_1(V)$) and a homomorphism $\mu : G \to \mathbb{R}$, yielding a G-action on $\widetilde{V}$ via $g \cdot (t, x) = (t + \mu(g), g \cdot x)$, and $V = (\mathbb{L}^1 \times \widetilde{M})/G$. Does the nature of $\hat{\partial}(V)$ depend on μ, or is that independent of μ? Under what circumstances is $\widehat{V}$ (aside from i^+) a simple product over a completion of M?

If V has a chronological group action from G and both V and V/G have only spacelike boundaries, is it necessary to assume $\widehat{V}/G$ is past-distinguishing in order to have $\hat{\partial}(V/G) \cong \hat{\partial}(V)/G$, or does that come automatically with spacelike boundaries?

Suppose V is strongly causal and has a foliation $\mathcal{F}$ by timelike curves, such that in every 2-sheet $S \subset V$ ruled by $\mathcal{F}$, each ruling enters the future and past of every point of S (treated as a spacetime in its own right). Then can one make any determination of $\hat{\partial}(V)$? If V is globally hyperbolic, how closely related is $\hat{\partial}(V)$ to the Cauchy surfaces?

And the Vague Conjecture of section 1: Can one read information on the invariance of spatial topology in V from the nature of its future and past causal boundaries?

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